# ON MATRICES ARISING IN THE FINITE FIELD ANALOGUE OF EULER'S INTEGRAL TRANSFORM

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ABSTRACT. Here we consider a special family of matrices which arise naturally in J. Greene's analogue of the Euler integral transform for finite field hypergeometric series defined in his 1984 Ph.D. thesis. We prove a conjecture of Ono concerning the determinants and eigenspaces of these special matrices.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

In his 1984 Ph.D. thesis [2], Greene initiated the study of hypergeometric functions over finite fields which are in many ways similar to the classical hypergeometric functions of Gauss. To define these functions, first let $A$ and $B$ be two multiplicative, complex-valued characters of $\mathbb{F}_q^\times$ extended to $\mathbb{F}_q$ by $A(0) = B(0) = 0$ and let $\binom{A}{B}$ be the normalized Jacobi sum

$$\binom{A}{B} := \frac{B(-1)}{q} J(A, \overline{B}) = \frac{B(-1)}{q} \sum_{x \in \mathbb{F}_q} A(x)\overline{B}(1-x). \tag{1}$$

Here $\overline{B}$ denoted that complex conjugate of $B$. Greene defined the *Gaussian hypergeometric function* ${}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \dots, & A_n \\ & B_1, & \dots, & B_n \end{matrix} \middle| x\right)_p$ by

$${}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \dots, & A_n \\ & B_1, & \dots, & B_n \end{matrix} \middle| x\right)_p := \frac{q}{q-1} \sum_{\chi} \binom{A_0\chi}{\chi} \binom{A_1\chi}{B_1\chi} \cdots \binom{A_0\chi}{B_n\chi} \chi(x).$$

Here $\sum_\chi$ denotes the some over all characters of $\mathbb{F}_q$. These functions have deep connections to certain combinatorial congruences of modular forms, as well as traces of Hecke operators and counting points on certain modular varieties [5]. For example, if we let ${}_2E_1(\lambda): \quad y^2 = x(x-1)(x-\lambda)$ be the Legendre form elliptic curve $(\lambda \neq 0, 1)$, we have the following result whenever $p \geq 5$ is a prime and $\lambda \in \mathbb{Q} - \{0, 1\}$ satisfies $\mathrm{ord}_p(\lambda(\lambda - 1)) = 0$ [4]:

$${}_2F_1\left(\begin{matrix} \phi_p & \phi_p \\ & \epsilon \end{matrix} \middle| \lambda\right)_p = -\frac{\phi_p(-1) \cdot {}_2a_1(p; \lambda)}{q}.$$

Here $\phi_p$ is the Legendre symbol modulo $p$, $\epsilon$ is the trivial character, and ${}_2a_1(p;\lambda)$ is the trace of Frobenius of ${}_2E_1(\lambda)$ at $p$. In analogy with the Euler integral transform for classical hypergeometric functions, it turns out that these Gaussian hypergeometric functions are traces of Gaussian hypergeometric functions of lower degree. More precisely, Greene proved the following fact:

$$(2)\quad {}_{n+1}F_n\left(\begin{matrix}A_0, & A_1, & \ldots, & A_n\\ & B_1, & \ldots, & B_n\end{matrix}\middle|\, x\right)_p$$
$$= \frac{A_nB_n(-1)}{p}\sum_{y=0}^{p-1} {}_nF_{n-1}\left(\begin{matrix}A_0, & A_1, & \ldots, & A_{n-1}\\ & B_1, & \ldots, & B_{n-1}\end{matrix}\middle|\, x\right)_p \cdot A_n(y)\overline{A}_nB_n(1-y).$$

This transform is related to the modularity of other varieties as well. For example, Ahlgren and Ono relate special values of ${}_4F_3$ hypergeometric functions to the coefficients of modular forms using the modularity of a certain Calabi-Yau threefold [1]. Thus, it is natural to consider the following matrix which plays the role of Euler's integral transform in the special case $A_i = \phi_q$ and $B_i = \epsilon$.

**Definition.** Let $p$ be an odd prime. Let $q = p^n \geq 5$ and $M_q$ be the $(q-2)\times(q-2)$ matrix $(a_{ij})$ indexed by $i, j \in \mathbb{F}_q - \{0,1\}$ where

$$a_{ij} = \phi_q(1-ij)\phi_q(ij).$$

Here $\phi_q$ denotes the quadratic character in $\mathbb{F}_q$. Based on numerical data, Ono made the following conjecture.

**Conjecture** (Ono)**.** *Let $f_q$ be the characteristic polynomial of $M_q$. Then*

$$f_q(x) = \begin{cases}(x+1)(x-1)(x+2)(x^2-q)^{(q-5)/2} & \text{if } \phi_q(-1)=1\\ x(x^2-3)(x^2-q)^{(q-5)/2} & \text{if } \phi_q(-1)=-1.\end{cases}$$

Our main result is the following.

**Theorem 1.1.** *Ono's conjecture is true.*

**Remarks.**
(1) *For the eigenvalues $0, -1, +1, -2$, we give explicit formulas for the eigenvectors (cf. Proposition 2.1).*
(2) *It is clear that the proof can be generalized to other cases of the analogue of Euler's integral formula. However, it is difficult to precisely state a more general theorem due to the absence of closed formulas for more general Jacobi sums.*

The paper is organized as follows. In §2 we establish the claimed formulas for the eigenvalues $\lambda \in \{\pm 1, -2\}$ using Jacobi sums. In §3 we complete the proof of the main theorem be proving that $(x^2-q)^{q-5}$ divides the characteristic polynomial of $M_q$ and that $x^2-3$ divides the characteristic polynomial when $\phi_q(-1) = -1$.

## 2. Eigenvectors for $\lambda \in \{0, \pm 1, -2\}$

The claimed formulas for the eigenvectors can be deduced using the following well-known lemma which we prove for completion.

**Lemma 1.** *If $a_0, a_1, a_2 \in \mathbb{F}_q$ and $a_2 \neq 0$, then*

$$\sum_{x \in \mathbb{F}_q} \phi(a_0 + a_1 x + a_2 x^2) = \begin{cases} -\phi(a_2) & \text{if } a_1^2 \neq 4a_0a_2 \\ \phi(a_2)(q-1) & \text{if } a_1^2 = 4a_0a_2. \end{cases}$$

*Proof.* Factor out $a_2$ and complete the square to get

$$\sum_{x \in \mathbb{F}_q} \phi(a_0 + a_1 x + a_2 x^2) = \phi(a_2) \sum_{x \in \mathbb{F}_q} \phi((x-a)^2 - b) = \phi(a_2) \sum_{x \in \mathbb{F}_q} \phi(x^2 - b),$$

where $a = -\frac{a_1}{2a_2}$ and $b = \frac{a_1^2 - 4a_0a_2}{4a_2}$. Then $b = 0$ if and only if the discriminant is 0, in which case the sum is clearly $\phi(a_2)(q-1)$. If $b \neq 0$, then the change of variables $y = x^2 - b$ gives

$$\sum_{x \in \mathbb{F}_q} \phi(x^2 - b) = \sum_y \phi(y)(\phi(y+b) + 1) = \sum_y \phi(y)\phi(y+b).$$

Now replacing $y$ by $\frac{b}{2}(y-1)$ and making the change of variables $z = 1 - y^2$ shows that

$$\sum_y \phi(y^2 + by) = \sum_y \phi(y^2 - 1) = \phi(-1) \sum_z \phi(z)(\phi(1-z) + 1) = \phi(-1) J(\phi, \phi) = -1.$$

This follows from the classical evaluation of $J(\phi, \phi)$ (for example, see [3]). □

We are in position to prove the first case of Theorem 1.1 when $\lambda \in \{0, \pm 1, -2\}$.

**Proposition 2.1.** *If $\phi_q(-1) = 1$, then $\lambda \in \{\pm 1, -2\}$ are eigenvalues for the matrices $M_q$. If $\phi_q(-1) = -1$, then $\lambda = 0$ is an eigenvector for $M_q$. These eigenvalues have the following corresponding eigenvectors $v = (v_k)_{k \in \mathbb{F}_q - \{0,1\}}$:*

$$\begin{aligned} \lambda &= -1, & v_k &= \phi_q(k)(\phi_q(k) - 1), \\ \lambda &= +1, & v_k &= 2\phi_q(k)(\phi_q(k-1) - \phi_q(k) - 1), \\ \lambda &= -2, & v_k &= \phi_q(k)(\phi_q(k-1) + \phi_q(k) + 1, \\ \lambda &= 0, & v_k &= \phi_q(k)(\phi_q(k) - \phi_q(k-1) + 1). \end{aligned}$$

*Proof.* We will give the full calculation for the eigenvalue $\lambda = -1$ when $\phi(-1) = 1$. The other three cases follow similarly.

When $\lambda = 1$, we must check the formula

$$-v_k = \sum_{s \neq 0,1} \phi(1 - ks)\phi(ks)v_s.$$

This is equivalent to showing that for $v'_k := \phi(k) - 1$, we have

$$-v'_k = \sum_{s\neq 0,1} \phi(1-ks)v'_s.$$

Using the lemma, we have

$$\begin{aligned} \sum_{s\neq 0,1} \phi(1-ks)\phi(s) - \sum_{s\neq 0,1} \phi(1-ks) &= -\phi(-k) - \phi(1-k) + \phi(1) + \phi(1-k) \\ &= -\phi(k) + 1. \end{aligned}$$

□

## 3. Determining the $\pm\sqrt{3}$ and $\pm\sqrt{q}$ Eigenspaces

Here we complete the proof of Theorem 1.1 by computing the remaining eigenvalues. We begin with the $\pm\sqrt{3}$-eigenvalues when $\phi_q(-1) = -1$.

**Proposition 3.1.** *If $\phi_q(-1) = -1$, then the characteristic polynomial of $M_q$ is divisible by $(x^2 - 3)$.*

*Proof.* By a similar calculation as in the proof of the previous proposition, we find that $v = (v_k), v' = (v'_k)$ are eigenvectors with eigenvalue 3 for $M_q^2$, where

$$v_k := \phi(k) + 1, \qquad v'_k := 1 + \phi(k)\phi(k+1).$$

As the characteristic polynomial of $M_q$ is in $\mathbb{Z}[x]$, we find that $x^2 - 3$ divides the characteristic polynomial of $M_q$. □

We now finish the proof of Theorem 1.1.

**Proposition 3.2.** *The characteristic polynomial of $M_q$ is divisible by $(x^2 - q)^{\frac{p-5}{2}}$.*

*Proof.* We begin by defining the following matrix related to $M_q$. Let $p$, $q$ be as above. Let $\widetilde{M_q} = (\phi(1-ij))_{i,j\in\mathbb{F}_q}$ be a $q \times q$ matrix indexed by values of $\mathbb{F}_q$.

Note that by row reduction it suffices to prove that the characteristic polynomial of $\widetilde{M_q}$ is divisible by $(x^2 - q)^{\frac{p-1}{2}}$. Consider the matrix $\widetilde{M_q^2} = \left(\sum_{k\in\mathbb{F}_q} \phi(1-ik)\phi(1-jk)\right)_{i,j\in\mathbb{F}_q}$

For each $a \in F_q - \{0, -1\}$, let $V_a = (v_i)_{i\in\mathbb{F}_q}$ be a vector indexed by elements of $\mathbb{F}_q$ such that $v_a = 1$, $v_{-1} = -\phi(-a)$, and $v_i = 0$ for all $i \in \mathbb{F}_q - \{0, a\}$. Then if $(u_i) = \widetilde{M_q}^2 V_a$, we have

$$\begin{aligned} (u_i) &= \left(\sum_{j\in\mathbb{F}_q} v_j \sum_{k\in\mathbb{F}_q} \phi(1-ik)\phi(1-jk)\right) \\ &= \left(\sum_{k\in\mathbb{F}_q} \phi(1-ik)\phi(1-ak) - \phi(-a) \sum_{k\in\mathbb{F}_q} \phi(1-ik)\phi(1+k)\right). \end{aligned}$$

Since $a \neq 0, -1$, by Lemma 1 we find

$$\begin{aligned} u_0 &= 0, \\ u_a &= q - 1 + \phi(-a)^2 = q, \\ u_{-1} &= -\phi(-a) - \phi(-a)(q-1) = -q\phi(-a). \end{aligned}$$

For all other $i$, we have $u_i = \phi(ia) - \phi(-a)\phi(-i) = 0$. Hence $V_a$ is an eigenvector for $\widetilde{M}_q^2$ with eigenvalue $q$.

We may also define $V_0 = (v_i)$ so that $v_0 = 1$, and $v_i = 0$ for all other $i \in \mathbb{F}_q$. Then if $(u_i) = \widetilde{M_q}^2 V_0$, we have $u_0 = \sum_{k \in \mathbb{F}_q} \phi(1) = q$, and $u_i = \sum_{k \in \mathbb{F}_q} \phi(1 - ik) = 0$ for $i \neq 0$. Hence $V_0$ is also an eigenvector for the eigenvalue $q$. This gives us a total of $q-1$ linearly independent eigenvectors corresponding to the eigenvalue $q$. Each eigenvalue (counting multiplicities) of $\widetilde{M}_q^2$ is the square of an eigenvalue of $\widetilde{M}_q$. Thus, $\widetilde{M}_q$ has eigenvalues $\pm\sqrt{q}$ of multiplicities that sum to $q-1$ and so $M_q$ has eigenvalues $\pm\sqrt{q}$ of multiplicities summing to at least $q-5$. By Lemma 1, we have that $\text{Trace}(M_q) = -1 - \phi(-1)$. But we already know that the sum of all other eigenvalues is $-1 - \phi(-1)$. Hence, the multiplicities of the $\pm\sqrt{q}$ eigenvalues must be equal.

□

## References

1. Scott Ahlgren and Ken Ono, *A Gaussian hypergeometric series evaluation and Apéry number congruences*, J. Reine Angew. Math. **518** (2000), 187–212. MR 1739404 (2001c:11057)
2. John Robert Greene, *Character sum analogues for hypergeometric and generalized hypergeometric functions over finite fields*, ProQuest LLC, Ann Arbor, MI, 1984, Thesis (Ph.D.)–University of Minnesota. MR 2633473
3. Kenneth Ireland and Michael Rosen, *A classical introduction to modern number theory*, second ed., Graduate Texts in Mathematics, vol. 84, Springer-Verlag, New York, 1990. MR 1070716 (92e:11001)
4. Masao Koike, *Orthogonal matrices obtained from hypergeometric series over finite fields and elliptic curves over finite fields*, Hiroshima Math. J. **25** (1995), no. 1, 43–52. MR 1322601 (96b:11079)
5. Ken Ono, *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, CBMS Regional Conference Series in Mathematics, vol. 102, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2004. MR 2020489 (2005c:11053)

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