JENSEN POLYNOMIALS FOR HOLOMORPHIC FUNCTIONS

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Abstract. Recent work of Griffin, Ono, Rolen, and Zagier shows that the Jensen polynomials for the Riemann xi function converge to the Hermite polynomials under a suitable normalization. We generalize this result, proving that the normalized Jensen polynomials for a large class of genus zero or one entire functions converge either to the Hermite polynomials, or to a new class of polynomials which can be written as a confluent hypergeometric function.

1. Introduction and statement of results

Given a sequence $\{\alpha(n)\}_{n=0}^{\infty}$ of real numbers, define the *Jensen polynomial of degree d and shift n* for α to be

$$J_{\alpha}^{d,n}(X) := \sum_{j=0}^{d} \binom{d}{j} \alpha(n+j) X^j.$$

In [1], the authors showed that for arithmetic functions satisfying appropriate growth conditions, there is a renormalization of the Jensen polynomials which converges as n goes to infinity. Following [1], we define renormalized Jensen polynomials given by

$$\widehat{J}_{\alpha}^{d,n}(X) := \frac{\delta(n)^{-d}}{\alpha(n)} J_{\alpha}^{d,n}\left(\frac{\delta(n)X - 1}{E(n)}\right),$$

where $\{\delta(n)\}$ and $\{E(n)\}$ are appropriate sequences of positive real numbers depending on α. The following from from [1] describes the limiting behavior of the renormalized Jensen polynomials.

Theorem 1.1 (Theorem 8 of [1])**.** *Suppose that $\{\alpha(n)\}, \{E(n)\}$, and $\{\delta(n)\}$ are sequences of real numbers, with $E(n)$ and $\delta(n)$ positive, and with $\delta(n)$ tending to 0, and that $F(t) = \sum_{i=0}^{\infty} c_i t^i$ is a formal power series with complex coefficients. For a fixed $d \geq 1$, suppose that there are real sequences $\{C_0(n)\}, \ldots, \{C_d(n)\}$, with $\lim_{n\to+\infty} C_i(n) = c_i$ for $0 \leq i \leq d$, such that for $0 \leq j \leq d$ we have*

$$\frac{\alpha(n+j)}{\alpha(n)} E(n)^{-j} = \sum_{i=0}^{d} C_i(n)\, \delta(n)^i j^i + \mathrm{o}\big(\delta(n)^d\big) \text{ as } n \to +\infty. \tag{1.1}$$

Then we have

$$\lim_{n\to+\infty} \widehat{J}_{\alpha}^{d,n}(X) = H_{F,d}(X),$$

where the polynomials $H_{F,m}(X) \in \mathbb{C}[x]$ are defined either by the generating function $F(-t)\, e^{Xt} = \sum H_{F,m}(X)\, t^m/m!$ or in closed form by $H_{F,m}(X) := m! \sum_{k=0}^{m} (-1)^{m-k} c_{m-k}\, X^k/k!$.

This theorem was applied to the sequence $\{\gamma(n)\}$ of positive real numbers defined by the Taylor expansion

$$\sum_{j=0}^{\infty} \frac{\gamma(j)}{j!} \cdot z^{2j} =: 8\xi\left(\frac{1}{2} + z\right),$$

where it was shown that there is an appropriate choice of $\delta(n)$ and $E(n)$, along with the generating function $F(t) = e^{-t^2}$, such that the sequence $\gamma(n)$ satisfies (1.1). Thus, as $n \to \infty$, the renormalized Jensen polynomials $\widehat{J}_{\gamma}^{d,n}(X)$ converge to (a non-standard normalization of) the Hermite polynomials $H_d(X)$. The normalized Hermite polynomials used in [1] are defined as the orthogonal polynomials for the measure $\mu(X) = e^{-X^2/4}$, or more explicitly by the generating function

$$\sum_{d=0}^{\infty} H_d(X) \frac{t^d}{d!} = e^{t^2 - Xt}$$

and in closed form by $H_d(X) := \sum_{k=0}^{\lfloor d/2 \rfloor} \frac{(-1)^k d!}{k!(d-2k)!} \cdot X^{d-2k}$. Since the coefficients of the Jensen polynomials are real, and their roots must come in conjugate pairs. However, since the zeros of $H_d(X)$ are real and simple, the zeros of the Jensen polynomials $J_{\gamma}^{d,n}$, for sufficiently large n, have distinct real parts and are therefore real.

In this paper, we study a large class of sequences $\{\alpha(n)\}$, for which a similar process holds.

Definition 1.2. *We will say that $\{\alpha(n)\}$ is* governed *by a function $F(t) = \sum_{i=0}^{\infty} c_i t^i$ if there are functions $\delta : \mathbb{R}^+ \to \mathbb{R}$ (called the uniformizer) and $E : \mathbb{R}^+ \to \mathbb{R}$ (called the exponential component) and a sequence of real numbers $\{C_i(n)\}$ for each $i \geq 0$, with the following properties.*

(1) The function δ is positive for all $x > 0$, but $\lim_{x\to\infty} \delta(x) = 0$.
(2) The function E is positive for all $x > 0$.
(3) For each $i \geq 0$, $\lim_{n\to\infty} C_i(n) = c_i$.
(4) We have that

$$\frac{\alpha(n+j)}{\alpha(n)} E(n)^{-j} = \sum_{i=0}^{\infty} C_i(n) \delta(n)^i j^i. \tag{1.2}$$

Remark. *Note that if a sequence $\alpha(n)$ is governed by a function $F(t)$, then it satisfies Theorem 1.1 for any $d \geq 1$. Therefore, we have*

$$\lim_{n\to+\infty} \widehat{J}_{\alpha}^{d,n}(X) = H_{F,d}(X),$$

where the polynomials $H_{F,m}(X)$ are defined as in the conclusion of Theorem 1.1.

In [1], the authors used the fact that the coefficients $\gamma(n)$ are interpolated by some analytic function to show that they satisfy the conditions of Theorem 1.1. Many other important functions share this property, including $L-$functions and hypergeometric functions. In this paper we consider sequences $\{\alpha(n)\}$ which arise from the values of some smooth function

$\alpha(z)$ at nonnegative integers. Specifically, our main theorems analyze the case when $\alpha(z)$ is an entire function of genus zero or one whose zeros have their real part bounded above. In this situation, $\alpha(z)$ has the Hadamard product expansion

$$\alpha(z) = cz^k e^{az} \prod_n \left(1 - \frac{z}{z_n}\right) e^{z/z_n}, \tag{1.3}$$

where $a, c \in \mathbb{R}, k \geq 0$, and the z_n are the nonzero roots of $\alpha(z) = 0$ (counted according to multiplicity) satisfying:

(1) $\mathrm{Re}(z_1) \geq \mathrm{Re}(z_2) \geq \mathrm{Re}(z_3) \ldots$,
(2) $\sum_n |z_n|^{-2} < \infty$.

Furthermore, we require that $\alpha(z)$ be real-valued along the real axis. We show that the possible generalized Hermite polynomials that arise depend on whether $\alpha(z)$ has finitely or infinitely many zeros.

The case of infinitely many roots includes the sequence $\gamma(n)$ discussed above. In particular the exact formula for $\gamma(n)$ given in (13) of [1] has a leading term of $\frac{n!}{(2n)!}$. In order to extend γ analytically, we write this term as a ratio of Γ-functions, which gives zeros at every negative half-integer.

Theorem 1.3. *Let $\alpha(z)$ be an entire function of genus ≤ 1, real-valued on $\mathbb{R}_{\geq 0}$, with* infinitely many roots. *Furthermore, assume the real parts of the the roots are bounded above. Then the sequence $\{\alpha(n)\}_{n=0}^{\infty}$ is governed by $F(t) = e^{-t^2}$, with some uniformizer $\delta(z)$ and exponential component $E(z)$.*

From Theorem 1.1 we can immediately deduce the following.

Corollary 1.4. *Assume the hypotheses and notation of Theorem 1.3. Then, for any $d \geq 1$ we have*

$$\lim_{n\to\infty} \frac{\delta(n)^{-d}}{\alpha(n)} J_\alpha^{d,n}\left(\frac{\delta(n)X - 1}{E(n)}\right) = H_d(X),$$

and since the Hermite polynomials have real and simple zeros, it follows that for all but finitely many values of n, the Jensen polynomials $J_\alpha^{d,n}(X)$ have only real zeros.

Our second main result concerns entire functions of the type in Theorem 1.3, but with a finite number of zeros.

Theorem 1.5. *Let $\alpha(z)$ be an entire function of genus ≤ 1, real-valued on $\mathbb{R}_{\geq 0}$, with $r < \infty$ zeros, counting multiplicities (so that $\alpha(z) = P(z)e^{az}$ for some polynomial $P(z)$). Then the sequence $\{\alpha(n)\}_{n=0}^{\infty}$ is governed by $F(t) = (1+t)^r e^{-rt}$, with some uniformizer $\delta(z)$ and exponential component $E(z)$.*

Example 1. *An intuitive explanation of Theorem 1.5 can be given as follows: The most basic functions satisfying the theorem statement are the polynomials $\alpha(z) = z^r$, for some r. We will see that natural choices for $\delta(z)$ and $E(z)$ are*

$$\delta(z) = 1/n \quad \text{and} \quad E(n) = e^{1/n}.$$

In the RHS of (1.2), finding the function $F(t)$ is essentially equivalent to substituting $t = j\delta(n)$ and then taking $n \to \infty$. Doing this, we have

$$\frac{\alpha(n+j)}{\alpha(n)} E(n)^{-j} = \left(1 + \frac{j}{n}\right)^r e^{-rj/n} \to (1+t)^r e^{-rt}.$$

Motivated by the conclusion of Theorem 1.1, we define $g_{r,m}(X) \in \mathbb{R}[X]$ to be the generalized polynomials generated by $(1+t)^r e^{-rt}$, so that

$$(1-t)^r e^{Xt+rt} =: \sum_{m=0}^{\infty} g_{r,m}(X) \frac{t^m}{m!} \tag{1.4}$$

By expanding the left-hand side and collecting powers of t, we find that

$$\begin{aligned} g_{r,m}(X) &= m! \sum_{j=0}^{m} \binom{r}{j} \frac{(X+r)^{m-j}(-1)^j}{(m-j)!} \\ &= U(-m, 1-m+r, X+r) \end{aligned} \tag{1.5}$$

where U is the confluent hypergeometric function of the second kind. As shown in (4.2), these functions are exactly the (*un-normalized*) Jensen polynomials for a certain modification of the a J-Bessel function given in (4.1).

Next, we calculate the limiting Jensen polynomials for sequences $\{\alpha(z)\}$ which satisfy the hypotheses of Theorem 1.5.

Corollary 1.6. *Assume the hypotheses and notation of Theorem 1.5. Then, for any $d \geq 1$ we have*

$$\lim_{n\to\infty} \frac{\delta(n)^{-d}}{\alpha(n)} J_{\alpha}^{d,n}\left(\frac{\delta(n)X - 1}{E(n)}\right) = g_{r,d}(X).$$

The polynomials $g_{r,m}(X)$ are very different in some regards from the standard Hermite polynomials $H_d(X)$, but they have several interesting properties which mimic some of the known properties of $H_d(X)$.

Theorem 1.7. *For any nonnegative integer m, the polynomials $g_{r,m}(X)$ defined in (1.4) satisfy the following:*

(1) We have

$$\frac{\mathrm{d}}{\mathrm{d}\,X} g_{r,m}(X) = m\, g_{r,m-1}(X). \tag{1.6}$$

(2) If r is a nonnegative integer, then

$$g_{r,m}(X) = (X+r)^{m-r} g_{m,r}(X). \tag{1.7}$$

(3) If r is a nonnegative integer, then $g_{r,m}(X)$ has only real roots.

This needs attention too

Remark. *In [3], Farmer shows that the cosine function is a "universal attractor," and our results illustrate this principle in a more precise manner that gives the actual limiting polynomials.*

2. Higher Logarithmic Derivatives

To prove Theorems 1.3 and 1.5, we show that they are special cases of more general Theorems 2.1 and 2.2, respectively, which only require that the function $\alpha(z)$ be real-valued and infinitely differentiable on the positive real axis.

The conditions of Theorems 2.1 and 2.2 given below are based on properties of the higher logarithmic derivatives of $\alpha(z)$, given for $m \geq 1$ by

$$A_m(z) := \frac{\mathrm{d}^{m-1}}{\mathrm{d}\, z^{m-1}} \left(\frac{\alpha'(z)}{\alpha(z)} \right). \tag{2.1}$$

If

$$\alpha(z) = cz^k \, e^{az} \prod_n \left(1 - \frac{z}{z_n}\right) e^{z/z_n}$$

is of the Hadamard product form given in (1.3), we can write

$$A_m(z) = (-1)^{m-1}(m-1)! \left[\frac{k}{z^m} + \sum_n \frac{1}{(z-z_n)^m} \right] \tag{2.2}$$

for any $m \geq 2$.

Theorems 2.1 and 2.2 apply for functions whose higher logarithmic derivatives are well-behaved in a certain sense. In particular, we require that the limits $\lim_{n\to\infty} n^m A_m(n)$ for $m > 2$ are dominated in a certain way by the $m = 2$ case, $\lim_{n\to\infty} n^2 A_2(n)$.

We deduce Theorem 1.3 from the following result:

Theorem 2.1. *Let $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be an analytic function. For each $m \geq 1$ let $A_m(x)$ be as above. Suppose that the sequence $\{A_2(n)\}$ tends to 0 as $n \to \infty$, but that*

$$\lim_{n\to\infty} n^2 \, A_2(n) = \pm\infty.$$

Let $s_\alpha \in \{\pm 1\}$ be the sign of this limit. Suppose further that for each $m \geq 3$, we have

$$\lim_{n\to\infty} \frac{A_m(n)^2}{A_2(n)^m} = 0.$$

Then the sequence $\{\alpha(n)\}_{n=0}^{\infty}$ is governed by $F(t) = e^{s_\alpha t^2}$, with exponential component $E(n) = e^{A_1(n)} = e^{\frac{\alpha'(n)}{\alpha(n)}}$ and uniformizer $\delta(n) = \sqrt{\frac{|A_2(n)|}{2}}$.

We deduce Theorem 1.5 from the following more general result.

Theorem 2.2. *Let $\alpha(x)$ be an infinitely differentiable real-valued function on the positive real axis. Define $A_m(x)$ as in (2.1), and suppose that for each integer $m \geq 2$ the sequential limit*

$$\lim_{n\to\infty} n^m A_m(n)$$

exists and is finite.

Let $r := -\lim_{n\to\infty} n^2 A_2(n)$. *Then* $\{\alpha(n)\}_{n=0}^{\infty}$ *is governed by* $F(t) = (1+t)^r e^{-rt}$, *with exponential component* $E(n) = e^{A_1(n)} = e^{\frac{\alpha'(n)}{\alpha(n)}}$ *and uniformizer* $\delta(n) = 1/n$.

Remarks.

(1) If $\alpha(n)$ is a rational function of degree D, or the product of a rational function of degree D and the exponential of a linear function, then it satisfies the conditions of Theorem 2.2 with $r = D$.

(2) Theorem 1.5 follows as an immediate consequence of Theorem 2.2 and (2.2).

2.1. **Proof of Theorem 2.1.** Define a sequence $B_m(n)$ for each $m \geq 0$ such that

$$\frac{\alpha(n+j)}{\alpha(n)} E(n)^{-j} =: \sum_{m=0}^{\infty} B_m(n) j^m.$$

By considering the coefficient of j^m in the product of the two power series

$$\frac{\alpha(n+j)}{\alpha(n)} = \sum_{i=0}^{\infty} \frac{\alpha^{(i)}(n)}{i!\,\alpha(n)} j^i$$

and

$$E(n)^{-j} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(-j\frac{\alpha'(n)}{\alpha(n)}\right)^k,$$

we can write $B_m(n)$ as an expression made up of derivatives of $\alpha(n)$. Furthermore, if $\alpha(n) > 0$ for all n, we have the alternate expansion

$$\begin{aligned}\frac{\alpha(n+j)}{\alpha(n)} E(n)^{-j} &= \exp\left(\log\left(\frac{\alpha(n+j)}{\alpha(n)}\right) - A_1(n) j\right)\\ &= \sum_{i=1}^{\infty} \frac{1}{i!} \left(\sum_{m=2}^{\infty} \frac{A_m(n)}{m!} j^m\right)^i.\end{aligned}$$

It follows that

$$\sum_{m=1}^{\infty} B_m(n) j^m = \sum_{i=1}^{\infty} \frac{1}{i!} \left(\sum_{k=2}^{\infty} \frac{A_k(n)}{k!} j^k\right)^i, \tag{2.3}$$

even if we do not assume that $\alpha(n) > 0$ for all n.

Now define $C_m(n) = \frac{1}{\delta(n)^m} B_m(n)$ and expand 2.3 in powers of j, obtaining

$$\begin{aligned}C_m(n) &= \frac{1}{\delta(n)^m} \sum_{\substack{\lambda\vdash m\\ \lambda_1=0}} \frac{1}{(\ell_\lambda)!} \prod_{i=2}^{m} \left(\frac{A_i(n)}{i!}\right)^{\lambda_i}\\ &= \sum_{\substack{\lambda\vdash m\\ \lambda_1=0}} \frac{1}{(\ell_\lambda)!} \prod_{i=2}^{m} \left(\frac{A_i(n)}{i!\,\delta(n)^i}\right)^{\lambda_i} \qquad (2.4)\end{aligned}$$

where λ is a partition of m, ℓ_λ is the length of λ, and λ_i is the number of parts of λ of size i.

By hypothesis, the factor $\frac{A_i(n)}{i!\,\delta(n)^i}$ tends to 0 as $n \to \infty$ for $i \geq 3$. Since additionally $\frac{A_2(n)}{2!\,\delta(n)^2} \to s_\alpha$ as $n \to \infty$, it follows that if $m = 2k$ then

$$\lim_{n\to\infty} C_m(n) =: c_m = s_\alpha^k/k!,$$

and if $m = 2k+1$ then

$$\lim_{n\to\infty} C_m(n) =: c_m = 0.$$

Thus $F(t) = \sum_{m=0}^{\infty} c_m t^m = \exp(s_\alpha t^2)$ which completes the proof.

2.2. **Proof of Theorem 2.2.** We begin by analyzing the growth of the logarithmic derivatives $A_m(n)$, working inductively on $m \geq 2$ to show that

$$\lim_{n\to\infty} n^m A_m(n) = (-1)^{m-1}\,(m-1)!\,r. \tag{2.5}$$

The base case is $m = 2$, which is given. Assuming (2.5) holds for $m = k$ as the inductive hypothesis, we have

$$(-1)^{k-1}\,(k-1)!\,r = \lim_{n\to\infty} \frac{A_k(n)}{n^{-k}} = \lim_{n\to\infty} \frac{A_{k+1}(n)}{-k\,n^{-(k+1)}}$$

by L'Hôpital's rule, which is valid since $\lim_{n\to\infty} A_k(n) = 0$, and since the last limit exists by hypothesis. Equation (2.5) follows for all $m \geq 2$ by induction.

For each $m \geq 0$, define $c_m(r) := \sum_{i=0}^{m} \frac{(-r)^i}{i!}\binom{r}{m-i}$, or by the generating function

$$(1+t)^r e^{-rt} =: \sum_{m=0}^{\infty} c_m(r) t^m. \tag{2.6}$$

We also have

$$\begin{aligned}(1+t)^r e^{-rt} &= \exp\left(r\log(1+t) - rt\right)\\ &= \sum_{k=0}^{\infty} \left(-r\sum_{i=2}^{\infty} \frac{(-t)^i}{i}\right)^k /k!\\ &= \sum_{m=0}^{\infty} \left[(-1)^m \sum_{\substack{\lambda\vdash m\\ \lambda_1=0}} \frac{(-r)^{\ell_\lambda}}{(\ell_\lambda)!} \prod_{i=2}^{m} \left(\frac{1}{i}\right)^{\lambda_i}\right] t^m,\end{aligned}$$

where we use the same partition notation as in (2.4). It follows that

$$(-1)^m \sum_{\substack{\lambda\vdash m\\ \lambda_1=0}} \frac{(-r)^{\ell_\lambda}}{(\ell_\lambda)!} \prod_{i=2}^{m} \left(\frac{1}{i}\right)^{\lambda_i} = c_m(r). \tag{2.7}$$

Define a sequence $C_m(n)$ for each $m \geq 0$ such that

$$\frac{\alpha(n+j)}{\alpha(n)} E(n)^{-j} =: \sum_{m=0}^{\infty} C_m(n)\delta(n)^m j^m;$$

then, to prove theorem 2.2, it suffices to show that $\lim_{n\to\infty} C_m(r) = c_m(r)$ for each $m \geq 0$. As in (2.4), we have

$$C_m(n) = \sum_{\substack{\lambda \vdash m \\ \lambda_1 = 0}} \frac{1}{(\ell_\lambda)!} \prod_{i=2}^{m} \left(\frac{A_i(n)}{i!\, \delta(n)^i} \right)^{\lambda_i}.$$

Then

$$\begin{aligned} \lim_{n\to+\infty} C_m(n) &= \sum_{\substack{\lambda \vdash m \\ \lambda_1 = 0}} \frac{1}{(\ell_\lambda)!} \prod_{i=2}^{m} \left(\frac{(-1)^{i-1}\, r}{i} \right)^{\lambda_i} \\ &= c_m(r) \end{aligned}$$

by (2.5) and (2.7). Theorem 2.2 follows.

3. Proof of Theorem 1.3

To prove Theorem 1.3, we show that all functions $\alpha(z)$ of the form given in (1.3) satisfy the hypotheses of Theorem 2.1, with $s_\alpha = -1$. Suppose that

$$\alpha(z) = cz^k\, e^{az} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{z/z_n}$$

is such a function, and consider only those values of $z = x \in \mathbb{R}^+$.

Recall that

$$A_m(x) = (-1)^{m-1}(m-1)! \left[\frac{k}{x^m} + \sum_{n=1}^{\infty} \frac{1}{(x - z_n)^m} \right].$$

For all $x > \max(0, 2\,\mathrm{Re}(z_1))$, we have $|x - z_n| > |z_n|$, so the convergence of $\sum_n |z_n|^{-2}$ implies that $A_m(x)$ is finite for all $m \geq 2$ and $x > \max(0, 2\,\mathrm{Re}(z_1))$. We can also deduce that each $A_m(x)$ (and in particular $A_2(x)$) tends to 0 as $x \to \infty$.

However,

$$x^2 A_2(x) = -\left[k + \sum_{n=1}^{\infty} \frac{1}{(1 - z_n/x)^2} \right]$$

tends to $-\infty$ as $x \to \infty$ because each term in the sum approaches 1.

Thus, to prove Theorem 1.3, it suffices just to show that

$$\lim_{x\to+\infty} \frac{A_m(x)^2}{A_2(x)^m} = 0 \tag{3.1}$$

for all $m \geq 3$.

Renumber the zeros of $\alpha(z)$ as $w_1, w_2, \ldots$ to include those which are equal to 0 (included k times), but maintaining the property $\mathrm{Re}(w_1) \geq \mathrm{Re}(w_2) \geq \ldots$. Fix $m \geq 3$ and define

$$f(x) := \frac{(-1)^m}{((m-1)!)^2} \frac{A_m(x)^2}{A_2(x)^m} = \frac{\left(\sum_{n=1}^{\infty} \frac{1}{(x-w_n)^m}\right)^2}{\left(\sum_{n=1}^{\infty} \frac{1}{(x-w_n)^2}\right)^m}.$$

We wish to show that $\lim_{x\to+\infty} f(x) = 0$. Let $\varepsilon > 0$; then we will show that there exists $X \in \mathbb{R}^+$ such that $f(x) < \varepsilon$ for all real $x > X$ (we have $f(x) > 0$ for $x > \mathrm{Re}(w_1)$).

For a natural number N, define

$$f_N(x) := \frac{\left(\sum_{n=1}^{N} \frac{1}{(x-w_n)^m}\right)^2}{\left(\sum_{n=1}^{N} \frac{1}{(x-w_n)^2}\right)^m} = \frac{\left(\sum_{n=1}^{N} \frac{1}{(1-w_n/x)^m}\right)^2}{\left(\sum_{n=1}^{N} \frac{1}{(1-w_n/x)^2}\right)^m}.$$

We have $\lim_{x\to\infty} f_N(x) = N^{2-m}$, since each term approaches 1 as $x \to \infty$ in the sums in both the numerator and denominator of the last quotient. Choose $N_0 \in \mathbb{N}$ such that $N_0^{2-m} < \frac{\varepsilon}{2}$. Then there exists $X \in \mathbb{R}^+$ such that $|f_{N_0}(x) - N_0^{2-m}| < \frac{\varepsilon}{2}$ for $x > X$. We will show that $f_N(x) > f(x)$ for all $N \in \mathbb{N}$ and $x \in \mathbb{R}^+$; then for $x > X$ we will have

$$\begin{aligned} f(x) &< f_{N_0}(x) \\ &\leq |f_{N_0}(x) - N_0^{2-m}| + N_0^{2-m} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

which will complete the proof.

To show that $f_N(x) > f(x)$, we instead show that $f_K(x) > f_{K+1}(x)$ for all $K \in \mathbb{N}$ and $x > \mathrm{Re}(w_1)$. Given $K \in \mathbb{N}$, renormalize

$$f_N(x) = \frac{\left(\sum_{n=1}^{N} \left(\frac{x-w_{K+1}}{x-w_n}\right)^m\right)^2}{\left(\sum_{n=1}^{N} \left(\frac{x-w_{K+1}}{x-w_n}\right)^2\right)^m} =: \frac{P_N(x)^2}{Q_N(x)^m}.$$

We have $P_N(x) \geq Q_N(x) \geq 1$. Then $f_{K+1}(x) = \frac{(P_K(x)+1)^2}{(Q_K(x)+1)^m}$, so $f_K(x) > f_{K+1}(x)$ is equivalent to

$$\frac{P_K(x)^2}{Q_K(x)^m} > \frac{(P_K(x)+1)^2}{(Q_K(x)+1)^m},$$

or

$$P_K(x)^2 \left(\sum_{i=0}^{m-1} \binom{m}{i} Q_K(x)^i\right) > Q_K(x)^m (2P_K(x) + 1).$$

We have

$$\begin{aligned} P_K(x)^2 \left(\sum_{i=0}^{m-1} \binom{m}{i} Q_K(x)^i\right) &> mP_K(x)^2 Q_K(x)^{m-1} \\ &\geq mP_K(x) Q_K(x)^m \\ &\geq Q_K(x)^m (2P_K(x) + 1) \end{aligned}$$

as desired. Theorem 1.3 follows.

4. The polynomials $g_{r,m}(X)$

We conclude the paper by proving the properties of the polynomials $g_{r,m}(X)$ contained in Theorem 1.7.

Proof of Theorem 1.7. Equations (1.6) and (1.7) follow easily from (1.5).

We now show that $g_{r,m}(X)$ has only real zeros for nonnegative integers r. If $m > r$, we use (1.7) to reduce to the case that $r \geq m$. In this case, we will show that the $g_{r,m}(X)$ are the (unnormalized) Jensen polynomials for a function constructed from the J-Bessel function. **Add reference for this** Since the J-Bessel function has real zeros and Weierstrass genus zero, we may apply a theorem of Jensen to show that these Jensen polynomials themselves have real zeros.

Let

$$G_{r,m}(X) := r!(-1)^m X^{\frac{m-r}{2}} J_{r-m}(2\sqrt{X}) = \sum_{n=0}^{\infty} \frac{r!}{(r-m+n)!}(-1)^{m-n}\frac{X^n}{n!} \tag{4.1}$$

and define the sequence $\{\alpha_{r,m}(n)\}$ to be the Taylor coefficients given by

$$\alpha_{r,m}(n) = (-1)^{m-n}\frac{r!}{(r-m+n)!}.$$

Then we see the Jensen polynomial $J_{\alpha}^{m,0}(X)$ satisfies

$$J_{\alpha}^{m,0}(X) = \sum_{j=0}^{m} \frac{r!}{(r-m+j)!}\binom{m}{j}(-1)^{m-j}X^j = g_{r,m}(X-r). \tag{4.2}$$

The J-Bessel functions have real zeros for $\nu > -1$, which implies that $G_{r,m}(X)$ has real zeros as well. Therefore, applying Jensen's theorem, we obtain the result. ∎

Finish citations

References

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